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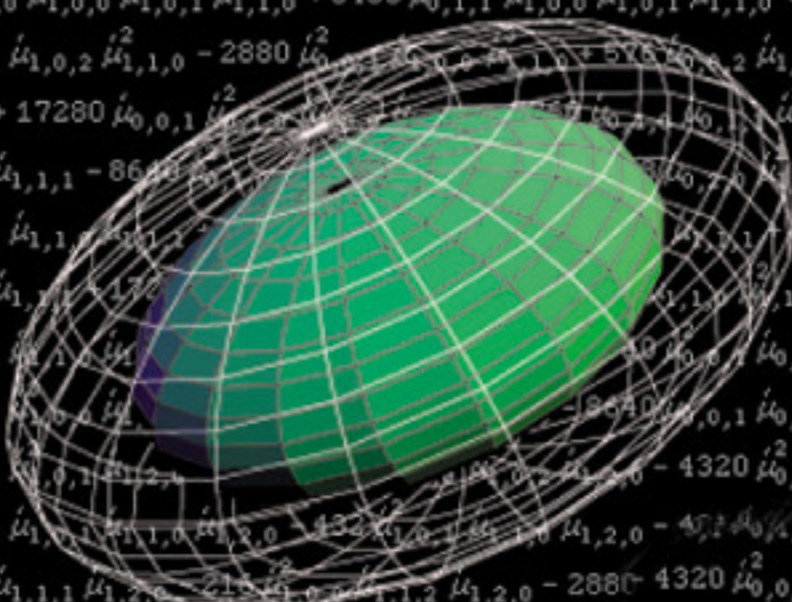

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Mathematical Statistics with *Mathematica*

Chapter 7 – Moments of Sampling Distributions

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Chapter 7

Moments of Sampling Distributions

7.1 Introduction

7.1 A Overview

Let $(X_1, \ldots, X_n)$ denote a random sample of size n drawn from a population random variable X. We can then distinguish between *population moments*:

$\acute{\mu}_r = E[X^r]$ raw moment of the *population*

$\mu_r = E[(X-\mu)^r]$ central moment of the *population*, where $\mu = E[X]$

and *sample moments*:

$\acute{m}_r = \frac{1}{n}\sum_{i=1}^{n} X_i^r$ *sample* raw moment

$m_r = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X})^r$ *sample* central moment, where $\overline{X} = \acute{m}_1$

where r is a positive integer. A *statistic* is a function of $(X_1, \ldots, X_n)$ that does not depend on any unknown parameter. Given this terminology, this chapter addresses two topics:

(i) *Unbiased estimators of population moments*
Given a random sample $(X_1, \ldots, X_n)$, we want to find a statistic that is an unbiased estimator of an unknown population moment. For instance, we might want to find unbiased estimators of population raw moments $\acute{\mu}_r$, or of central moments μ_r, or of cumulants κ_r. We might even want to find unbiased estimators of products of population moments such as $\mu_2\,\mu_4$. These problems are discussed in §7.2.

(ii) *Population moments of sample moments*
Because $(X_1, \ldots, X_n)$ is a collection of random variables, it follows that statistics like $\acute{m}_r$ and m_r are themselves random variables, having their own distribution, and thus their own population moments. Thus, for instance, we may want to find the expectation of m_2. Since $E[m_2]$ is just the first raw moment of m_2, we can denote this problem by $\acute{\mu}_1(m_2)$. Similarly, $\mathrm{Var}(\acute{m}_1)$ is just the second central moment of $\acute{m}_1$, so we can denote this problem by $\mu_2(\acute{m}_1)$. This is the topic of *moments of moments*, and it is discussed in §7.3.

7.1 B Power Sums and Symmetric Functions

Power sums are the *lingua franca* of this chapter. The r^{th} *power sum* is defined as

$$s_r = \sum_{i=1}^{n} X_i^r, \qquad r = 1, 2, \ldots \tag{7.1}$$

The sample raw moments can easily be expressed in terms of power sums:

$$\acute{m}_1 = \frac{s_1}{n}, \quad \acute{m}_2 = \frac{s_2}{n}, \quad \ldots, \quad \acute{m}_r = \frac{s_r}{n}. \tag{7.2}$$

One can also express the sample central moments in terms of power sums, and **mathStatica** automates these conversions.[1] Here, for example, we express the 2^{nd} sample central moment m_2 in terms of power sums:

```
SampleCentralToPowerSum[2]
```

$$m_2 \to -\frac{s_1^2}{n^2} + \frac{s_2}{n}$$

Next, we express $\acute{m}_3$ and m_4 in terms of power sums:

```
SampleRawToPowerSum[3]
```

$$\acute{m}_3 \to \frac{s_3}{n}$$

```
SampleCentralToPowerSum[4]
```

$$m_4 \to -\frac{3\, s_1^4}{n^4} + \frac{6\, s_1^2\, s_2}{n^3} - \frac{4\, s_1\, s_3}{n^2} + \frac{s_4}{n}$$

These functions also handle multivariate conversions. For instance, to express the bivariate sample central moment $m_{3,1} = \frac{1}{n}\sum_{i=1}^{n}\left((X_i - \overline{X})^3\,(Y_i - \overline{Y})^1\right)$ into power sums, enter:

```
SampleCentralToPowerSum[{3, 1}]
```

$$m_{3,1} \to -\frac{3\, s_{0,1}\, s_{1,0}^3}{n^4} + \frac{3\, s_{1,0}^2\, s_{1,1}}{n^3} + \frac{3\, s_{0,1}\, s_{1,0}\, s_{2,0}}{n^3} - \frac{3\, s_{1,0}\, s_{2,1}}{n^2} - \frac{s_{0,1}\, s_{3,0}}{n^2} + \frac{s_{3,1}}{n}$$

where each bivariate power sum $s_{r,t}$ is defined by

$$s_{r,t} = \sum_{i=1}^{n} X_i^r\, Y_i^t. \tag{7.3}$$

For a multivariate application, see *Example 7*. Power sums are also discussed in §7.4.

A function $f(x_1, \ldots, x_n)$ is said to be *symmetric* if it is unchanged after any permutation of the x's; that is, if say $f(x_1, x_2, x_3) = f(x_2, x_1, x_3)$. Thus,

$$x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

is a symmetric function of $x_1, x_2, \ldots, x_n$. Examples of symmetric statistics include moments, product moments, h-statistics (h_r) and k-statistics (k_r). Symmetry is a most desirable property for an estimator to have: it generally amounts to saying that an estimate should *not* depend on the order in which the observations were made. The tools provided in this chapter apply to any rational, integral, algebraic symmetric function. This includes m_r, $m_r\,\mathrm{k}_r$ or $m_r + \mathrm{h}_r$, but not m_r / k_r nor $\sqrt{m_r}$. Symmetric functions are also discussed in more detail in §7.4.

7.2 Unbiased Estimators of Population Moments

On browsing through almost any statistics textbook, one encounters an *estimator of population variance* defined by $\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$, where $\overline{X}$ is the sample mean. It is only natural to ponder why the denominator in this expression is $n-1$ rather than n. The answer is that $n-1$ yields an unbiased estimator of the population variance, while n yields a biased estimator. This section provides a toolset to attack such questions, not only for the variance, but for any population moment. We introduce h-statistics which are unbiased estimators of central population moments, and k-statistics which are unbiased estimators of population cumulants, and then generalise these statistics to encompass products of moments as well as multivariate moments. To do so, we couch our language in terms of power sums (see §7.1 B), which are closely related to sample moments. Although we assume an infinite universe, the results do extend to finite populations. For the finite univariate case, see Stuart and Ord (1994, Section 12.20); for the finite multivariate case, see Dwyer and Tracy (1980).

7.2 A Unbiased Estimators of Raw Moments of the Population

By the fundamental expectation result (7.15), it can be shown that sample raw moments $\acute{m}_r$ are unbiased estimators of population raw moments $\acute{\mu}_r$. That is,

$$E\left[\acute{m}_r\right] = \acute{\mu}_r . \tag{7.4}$$

However, products of sample raw moments are *not* unbiased estimators of products of population raw moments. For instance, $\acute{m}_2\,\acute{m}_3$ is not an unbiased estimator of $\acute{\mu}_2\,\acute{\mu}_3$. Unbiased estimators of products of raw moments are discussed in *Example 6* and in §7.4 A.

7.2 B h-statistics: Unbiased Estimators of Central Moments

The h-statistic h_r is an unbiased estimator of μ_r, defined by

$$E[\mathrm{h}_r] = \mu_r . \tag{7.5}$$

That is, h_r is the statistic whose expectation is the central moment μ_r. Of all unbiased estimators of μ_r, the h-statistic is the only one that is symmetric. Halmös (1946) showed that not only is h_r unique, but its variance $\mathrm{Var}(h_r) = E[(h_r - \mu_r)^2]$ is a minimum relative to all other unbiased estimators. We express h-statistics in terms of power sums, following Dwyer (1937) who introduced the term h-statistic. Here are the first four h-statistics:

```
Table[HStatistic[i], {i, 4}] // TableForm
```

$$h_1 \to 0$$
$$h_2 \to \frac{-s_1^2 + n\, s_2}{(-1+n)\, n}$$
$$h_3 \to \frac{2\, s_1^3 - 3\, n\, s_1\, s_2 + n^2\, s_3}{(-2+n)\,(-1+n)\, n}$$
$$h_4 \to \frac{-3\, s_1^4 + 6\, n\, s_1^2\, s_2 + (9-6\, n)\, s_2^2 + (-12+8\, n-4\, n^2)\, s_1\, s_3 + (3\, n-2\, n^2+n^3)\, s_4}{(-3+n)\,(-2+n)\,(-1+n)\, n}$$

If we express the results in terms of sample central moments m_i, they appear neater:

```
Table[HStatisticToSampleCentral[i], {i, 4}] // TableForm
```

$$h_1 \to 0$$
$$h_2 \to \frac{n\, m_2}{-1+n}$$
$$h_3 \to \frac{n^2\, m_3}{(-2+n)\,(-1+n)}$$
$$h_4 \to \frac{(9-6\, n)\, n^2\, m_2^2 + n\,(3\, n-2\, n^2+n^3)\, m_4}{(-3+n)\,(-2+n)\,(-1+n)\, n}$$

⊕ ***Example 1:*** Unbiased Estimator of the Population Variance

We wish to find an unbiased estimator of the population variance μ_2. It follows immediately that an unbiased estimator of μ_2 is h_2. Here is h_2 expressed in terms of sample central moments:

```
HStatisticToSampleCentral[2]
```

$$h_2 \to \frac{n\, m_2}{-1 + n}$$

which is identical to the standard textbook result $\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$. Given that $\frac{n}{n-1}\, m_2$ is an unbiased estimator of population variance, it follows that m_2 is a biased estimator of population variance; §7.3 provides a toolset that enables one to calculate $E[m_2]$, and hence measure the bias. ■

⊕ ***Example 2:*** Unbiased Estimator of μ_5 when $n = 11$

If the sample size n is known, and $n > r$, the function `HStatistic[r, n]` returns h_r. When $n = 11$, h_5 is:

```
HStatistic[5, 11]
```

$$h_5 \to \frac{4\, s_1^5 - 110\, s_1^3\, s_2 + 270\, s_1\, s_2^2 + 850\, s_1^2\, s_3 - 990\, s_2\, s_3 - 4180\, s_1\, s_4 + 9196\, s_5}{55440}$$

⊕ ***Example 3:*** Working with Data

The following data is a random sample of 30 lightbulbs, recording the observed life of each bulb in weeks:

```
data = {16.34, 10.76, 11.84, 13.55, 15.85, 18.20,
   7.51, 10.22, 12.52, 14.68, 16.08, 19.43,
   8.12, 11.20, 12.95, 14.77, 16.83, 19.80,
   8.55, 11.58, 12.10, 15.02, 16.83, 16.98,
   19.92, 9.47, 11.68, 13.41, 15.35, 19.11};
```

We wish to estimate the third central moment μ_3 of the population. If we simply calculated m_3 (a biased estimator), we would get the following estimate:

```
<< Statistics`

CentralMoment[data, 3]

-1.30557
```

By contrast, h_3 is an *unbiased* estimator. Evaluating the power sums $s_r = \sum_{i=1}^{n} X_i^r$ yields:

```
HStatistic[3, 30] /. s_r_ :→ Plus @@ data^r

h3 → -1.44706
```

mathStatica's `UnbiasedCentralMoment` function automates this process, making it easier to use. That is, `UnbiasedCentralMoment[`*data*, *r*`]` estimates μ_r using the unbiased estimator h_r. Of course, it yields the same result:

```
UnbiasedCentralMoment[data, 3]

-1.44706
```

Chapter 5 makes frequent use of this function. ■

○ *Polyaches* (Generalised h-statistics)

The generalised h-statistic (Tracy and Gupta (1974)) is defined by

$$E[h_{\{r,s,\ldots,t\}}] = \mu_r \, \mu_s \cdots \mu_t \, . \tag{7.6}$$

That is, $h_{\{r,s,\ldots,t\}}$ is the statistic whose expectation is the product of the central moments $\mu_r \, \mu_s \cdots \mu_t$. Just as Tukey (1956) created the onomatopoeic term ‘polykay’ to denote the generalised k-statistic (discussed below), we neologise ‘polyache’ to denote the generalised h-statistic. Perhaps, to paraphrase Kendall, there really are limits to linguistic miscegenation that should not be exceeded ☺.[2] Note that the polyache of a single term `PolyH[{r}]` is identical to `HStatistic[r]`.

⊕ ***Example 4:*** Find an Unbiased Estimator of $\mu_2^2\,\mu_3$

The solution is the polyache $\text{h}_{\{2,2,3\}}$:

```
PolyH[{2, 2, 3}]
```

$$\text{h}_{\{2,2,3\}} \to (2\,s_1^7 - 7\,n\,s_1^5\,s_2 + (30 - 18\,n + 8\,n^2)\,s_1^3\,s_2^2 + (60 - 63\,n + 21\,n^2 - 3\,n^3)\,s_1\,s_2^3 + (-40 + 24\,n + n^2)\,s_1^4\,s_3 + (-120 + 96\,n - 24\,n^2 - 2\,n^3)\,s_1^2\,s_2\,s_3 + (-20\,n + 21\,n^2 - 7\,n^3 + n^4)\,s_2^2\,s_3 + (80 - 40\,n - 4\,n^2 + 4\,n^3)\,s_1\,s_3^2 + (60 - 8\,n - 12\,n^2)\,s_1^3\,s_4 + (-120 + 140\,n - 63\,n^2 + 13\,n^3)\,s_1\,s_2\,s_4 + (-20\,n + 10\,n^2 + n^3 - n^4)\,s_3\,s_4 + (48 - 92\,n + 30\,n^2 + 2\,n^3)\,s_1^2\,s_5 + (36\,n - 34\,n^2 + 12\,n^3 - 2\,n^4)\,s_2\,s_5 + (-28\,n + 42\,n^2 - 14\,n^3)\,s_1\,s_6 + (4\,n^2 - 6\,n^3 + 2\,n^4)\,s_7)\,/\,((-6+n)\,(-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\,n)$$

Because h-statistics are symmetric functions, the ordering of the arguments, $\text{h}_{\{2,3,2\}}$ versus $\text{h}_{\{2,2,3\}}$, does not matter:

```
PolyH[{2, 3, 2}]⟦2⟧ == PolyH[{2, 2, 3}]⟦2⟧
```

```
True
```

When using generalised h-statistics $\text{h}_{\{r,s,\ldots,t\}}$, the weight of the statistic can easily become quite large. Here, $\text{h}_{\{2,2,3\}}$ has weight $7 = 2 + 2 + 3$, and it contains terms such as s_7. Although $\text{h}_{\{2,2,3\}}$ is an unbiased estimator of $\mu_2^2\,\mu_3$, some care must be taken in small samples because the variance of the estimator may be large. Intuitively, the effect of an outlier in a small sample is accentuated by terms such as s_7. In this vein, *Example 11* compares the performance of $\text{h}_{\{2,2\}}$ to h_2^2. ■

7.2 C k-statistics: Unbiased Estimators of Cumulants

The k-statistic k_r is an unbiased estimator of κ_r, defined by

$$E[\text{k}_r] = \kappa_r, \qquad r = 1, 2, \ldots \tag{7.7}$$

That is, k_r is the (unique) symmetric statistic whose expectation is the r^{th} cumulant κ_r. From Halmös (1946), we again know that, of all unbiased estimators of κ_r, the k-statistic is the only one that is symmetric, and its variance $\text{Var}(\text{k}_r) = E[(\text{k}_r - \kappa_r)^2]$ is a minimum relative to all other unbiased estimators. Following Fisher (1928), we define k-statistics in terms of power sums. Here, for instance, are the first four k-statistics:

```
Table[KStatistic[i], {i, 4}] // TableForm
```

$$\text{k}_1 \to \frac{s_1}{n}$$

$$\text{k}_2 \to \frac{-s_1^2 + n\,s_2}{(-1+n)\,n}$$

$$\text{k}_3 \to \frac{2\,s_1^3 - 3\,n\,s_1\,s_2 + n^2\,s_3}{(-2+n)\,(-1+n)\,n}$$

$$\text{k}_4 \to \frac{-6\,s_1^4 + 12\,n\,s_1^2\,s_2 + (3\,n - 3\,n^2)\,s_2^2 + (-4\,n - 4\,n^2)\,s_1\,s_3 + (n^2 + n^3)\,s_4}{(-3+n)\,(-2+n)\,(-1+n)\,n}$$

Once again, if we express these results in terms of sample central moments m_i, they appear neater:

```
Table[KStatisticToSampleCentral[i], {i, 4}] // TableForm
```

$k_1 \to 0$

$k_2 \to \frac{n\, m_2}{-1+n}$

$k_3 \to \frac{n^2\, m_3}{(-2+n)\,(-1+n)}$

$k_4 \to \frac{n^2\,(3\,n-3\,n^2)\,m_2^2+n\,(n^2+n^3)\,m_4}{(-3+n)\,(-2+n)\,(-1+n)\,n}$

Stuart and Ord (1994) provide tables of k-statistics up to $r = 8$, though published results do exist to $r = 12$. Ziaud-Din (1954) derived k_9, and k_{10} (contains errors), Ziaud-Din (1959) derived k_{11} (contains errors), while Ziaud-Din and Ahmad (1960) derived k_{12}. The `KStatistic` function makes it simple to derive correct solutions 'on the fly', and it extends the analysis well past k_{12}. For instance, it takes just a few seconds to derive the 15^{th} k-statistic on our reference personal computer:

```
KStatistic[15]; // Timing
```

{2.8 Second, Null}

But beware — the printed result will fill many pages!

○ *Polykays* (Generalised k-statistics)

Dressel (1940) introduced the generalised k-statistic $k_{\{r,s,\ldots,t\}}$ (now also called polykay) defined by

$$E[k_{\{r,s,\ldots,t\}}] \;=\; \kappa_r\,\kappa_s\,\cdots\,\kappa_t\,. \tag{7.8}$$

That is, a polykay $k_{\{r,s,\ldots,t\}}$ is the statistic whose expectation is the product of the cumulants $\kappa_r\,\kappa_s\,\cdots\,\kappa_t$. Here is the polykay $k_{\{2,4\}}$ in terms of power sums:

```
PolyK[{2, 4}]
```

$$k_{\{2,4\}} \to (6\,s_1^6 - 18\,n\,s_1^4\,s_2 + (30 - 27\,n + 15\,n^2)\,s_1^2\,s_2^2 + (60 - 60\,n + 21\,n^2 - 3\,n^3)\,s_2^3 + (-40 + 36\,n + 4\,n^2)\,s_1^3\,s_3 + (-120 + 100\,n - 24\,n^2 - 4\,n^3)\,s_1\,s_2\,s_3 + (40 - 10\,n - 10\,n^2 + 4\,n^3)\,s_3^2 + (60 - 20\,n - 15\,n^2 - n^3)\,s_1^2\,s_4 + (-60 + 45\,n - 10\,n^2 + n^4)\,s_2\,s_4 + (24 - 42\,n + 12\,n^2 + 6\,n^3)\,s_1\,s_5 + (-4\,n + 7\,n^2 - 2\,n^3 - n^4)\,s_6) \,/\, ((-5+n)\,(-4+n)\,(-3+n)\,(-2+n)\,(-1+n)\,n)$$

Finally, note that the polykay of a single term `PolyK[{r}]` is identical to `KStatistic[r]`; however, they use different algorithms, and the latter function is more efficient computationally.

⊕ ***Example 5:*** Find an Unbiased Estimator of κ_2^2

Solution: The required unbiased estimator is the polykay $k_{\{2,2\}}$:

```
PolyK[{2, 2}]
```

$$k_{\{2,2\}} \to \frac{s_1^4 - 2\,n\,s_1^2\,s_2 + (3 - 3\,n + n^2)\,s_2^2 + (-4 + 4\,n)\,s_1\,s_3 + (n - n^2)\,s_4}{(-3 + n)\,(-2 + n)\,(-1 + n)\,n}$$

For the lightbulb data set of *Example 3*, this yields the estimate:

```
PolyK[{2, 2}, 30] /. s_r_ :→ Plus @@ data^r
```

$k_{\{2,2\}} \to 154.118$

By contrast, k_2^2 is a biased estimator of κ_2^2, and yields a different estimate:

```
k2 = KStatistic[2, 30] /. s_r_ :→ Plus @@ data^r
k2[[2]]^2
```

$k_2 \to 12.6501$

160.024

Example 11 compares the performance of the unbiased estimator $h_{\{2,2\}}$ to the biased estimator h_2^2 (note that $k_{\{2,2\}} = h_{\{2,2\}}$, and $k_2 = h_2$). ■

⊕ ***Example 6:*** Find an Unbiased Estimator of the Product of Raw Moments $\acute{\mu}_3\,\acute{\mu}_4$

Polykays can be used to find unbiased estimators of quite general expressions. For instance, to find an unbiased estimator of the product of raw moments $\acute{\mu}_3\,\acute{\mu}_4$, we may proceed as follows:

Step (i): Convert $\acute{\mu}_3\,\acute{\mu}_4$ into cumulants:

```
p = μ́3 μ́4 /. Table[RawToCumulant[i], {i, 3, 4}] // Expand
```

$$\kappa_1^7 + 9\,\kappa_1^5\,\kappa_2 + 21\,\kappa_1^3\,\kappa_2^2 + 9\,\kappa_1\,\kappa_2^3 + 5\,\kappa_1^4\,\kappa_3 + 18\,\kappa_1^2\,\kappa_2\,\kappa_3 + 3\,\kappa_2^2\,\kappa_3 + 4\,\kappa_1\,\kappa_3^2 + \kappa_1^3\,\kappa_4 + 3\,\kappa_1\,\kappa_2\,\kappa_4 + \kappa_3\,\kappa_4$$

Step (ii): Find an unbiased estimator of each term in this expression. Since each term is a product of cumulants, the unbiased estimator of each term is a polykay. The first term κ_1^7 becomes $k_{\{1,1,1,1,1,1,1\}}$, while $9\,\kappa_1^5\,\kappa_2$ becomes $9\,k_{\{1,1,1,1,1,2\}}$, and so on. While we could do all this manually, there is an easier way! If $p(x)$ is a symmetric polynomial in x, the **mathStatica** function `ListForm[p, x]` will convert p into a 'list form' suitable for use by `PolyK` and many other functions. Note that `ListForm` should *only* be called on polynomials that have just been expanded using `Expand`. The order of the terms is now reversed:

```
p1 = ListForm[p, κ]
```

```
κ[{3, 4}] + 3 κ[{1, 2, 4}] + 4 κ[{1, 3, 3}] + 3 κ[{2, 2, 3}] +
 κ[{1, 1, 1, 4}] + 18 κ[{1, 1, 2, 3}] + 9 κ[{1, 2, 2, 2}] +
 5 κ[{1, 1, 1, 1, 3}] + 21 κ[{1, 1, 1, 2, 2}] +
 9 κ[{1, 1, 1, 1, 1, 2}] + κ[{1, 1, 1, 1, 1, 1, 1}]
```

Replacing each κ term by `PolyK` yields the desired estimator:

```
p1 /. κ[x_] :→ PolyK[x]⟦2⟧ // Factor
```

$$\frac{s_3\, s_4 - s_7}{(-1+n)\, n}$$

which is surprisingly neat. *Example 15* provides a more direct way of finding unbiased estimators of products of raw moments, but requires some knowledge of augmented symmetrics to do so. ■

7.2 D Multivariate h- and k-statistics

The multivariate h-statistic $\mathrm{h}_{r,s,\ldots,t}$ is defined by

$$E[\mathrm{h}_{r,s,\ldots,t}] = \mu_{r,s,\ldots,t}\,. \tag{7.9}$$

That is, $\mathrm{h}_{r,s,\ldots,t}$ is the statistic whose expectation is the q-variate central moment $\mu_{r,s,\ldots,t}$ (see §6.2 B), where

$$\mu_{r,s,\ldots,t} = E\left[(X_1 - E[X_1])^r\, (X_2 - E[X_2])^s \cdots (X_q - E[X_q])^t\right] \tag{7.10}$$

Some care with notation is required here. We use curly brackets {} to distinguish between the multivariate h-statistics $\mathrm{h}_{r,s,\ldots,t}$ of this section and the univariate polyaches $\mathrm{h}_{\{r,s,\ldots,t\}}$ (generalised h-statistics) discussed in §7.2 B.

The **mathStatica** function `HStatistic[{`r, s, …, t`}]` yields the multivariate h-statistic $\mathrm{h}_{r,s,\ldots,t}$. Here are two bivariate examples:

```
HStatistic[{1, 1}]
HStatistic[{2, 1}]
```

$$\mathrm{h}_{1,1} \to \frac{-s_{0,1}\, s_{1,0} + n\, s_{1,1}}{(-1+n)\, n}$$

$$\mathrm{h}_{2,1} \to \frac{2\, s_{0,1}\, s_{1,0}^2 - 2\, n\, s_{1,0}\, s_{1,1} - n\, s_{0,1}\, s_{2,0} + n^2\, s_{2,1}}{(-2+n)\,(-1+n)\, n}$$

where each bivariate power sum $s_{r,t}$ is defined by

$$s_{r,t} = \sum_{i=1}^{n} X_i^r\, Y_i^t\,.$$

Higher variate examples soon become quite lengthy. Here is a simple trivariate example:

```
HStatistic[{2, 1, 1}]
```

$$h_{2,1,1} \to (-3\, s_{0,0,1}\, s_{0,1,0}\, s_{1,0,0}^2 + n\, s_{0,1,1}\, s_{1,0,0}^2 + 2\, n\, s_{0,1,0}\, s_{1,0,0}\, s_{1,0,1} + 2\, n\, s_{0,0,1}\, s_{1,0,0}\, s_{1,1,0} - 2\,(-3 + 2\, n)\, s_{1,0,1}\, s_{1,1,0} - 2\,(3 - 2\, n + n^2)\, s_{1,0,0}\, s_{1,1,1} + n\, s_{0,0,1}\, s_{0,1,0}\, s_{2,0,0} - (-3 + 2\, n)\, s_{0,1,1}\, s_{2,0,0} - (3 - 2\, n + n^2)\, s_{0,1,0}\, s_{2,0,1} - (3 - 2\, n + n^2)\, s_{0,0,1}\, s_{2,1,0} + n\,(3 - 2\, n + n^2)\, s_{2,1,1}) \,/\, ((-3 + n)\,(-2 + n)\,(-1 + n)\, n)$$

In similar fashion, the multivariate k-statistic $k_{r,s,\ldots,t}$ is defined by

$$E[k_{r,s,\ldots,t}] = \kappa_{r,s,\ldots,t}\,. \tag{7.11}$$

That is, $k_{r,s,\ldots,t}$ is the statistic whose expectation is the multivariate cumulant $\kappa_{r,s,\ldots,t}$. Multivariate cumulants were briefly discussed in §6.2 C and §6.2 D. Here is a bivariate result originally given by Fisher (1928):

```
KStatistic[{3, 1}]
```

$$k_{3,1} \to (-6\, s_{0,1}\, s_{1,0}^3 + 6\, n\, s_{1,0}^2\, s_{1,1} + 6\, n\, s_{0,1}\, s_{1,0}\, s_{2,0} - 3\,(-1 + n)\, n\, s_{1,1}\, s_{2,0} - 3\, n\,(1 + n)\, s_{1,0}\, s_{2,1} - n\,(1 + n)\, s_{0,1}\, s_{3,0} + n^2\,(1 + n)\, s_{3,1}) \,/\, ((-3 + n)\,(-2 + n)\,(-1 + n)\, n)$$

Multivariate polykays and multivariate polyaches are not currently implemented in **mathStatica**.

⊕ ***Example 7:*** American NFL Matches: Estimating the Central Moment $\mu_{2,1}$

The following data is taken from American National Football League games in 1986; see Csörgö and Welsh (1989). Variable X_1 measures the time from the start of the game until the first points are scored by kicking the ball between the end-posts (a field goal), while X_2 measures the time from the start of the game until the first points are scored by a touchdown. Times are given in minutes and seconds. If $X_1 < X_2$, the first score is a field goal; if $X_1 = X_2$, the first score is a converted touchdown; if $X_1 > X_2$, the first score is an unconverted touchdown:

```
data = {{2.03, 3.59}, {7.47, 7.47}, {7.14, 9.41},
   {31.08, 49.53}, {7.15, 7.15}, {4.13, 9.29}, {6.25, 6.25},
   {10.24, 14.15}, {11.38, 17.22}, {14.35, 14.35},
   {17.5, 17.5}, {9.03, 9.03}, {10.34, 14.17}, {6.51, 34.35},
   {14.35, 20.34}, {4.15, 4.15}, {15.32, 15.32}, {8.59, 8.59},
   {2.59, 2.59}, {1.23, 1.23}, {11.49, 11.49}, {10.51, 38.04},
   {0.51, 0.51}, {7.03, 7.03}, {32.27, 42.21}, {5.47, 25.59},
   {1.39, 1.39}, {2.54, 2.54}, {10.09, 10.09}, {3.53, 6.26},
   {10.21, 10.21}, {5.31, 11.16}, {3.26, 3.26}, {2.35, 2.35},
   {8.32, 14.34}, {13.48, 49.45}, {6.25, 15.05}, {7.01, 7.01},
   {8.52, 8.52}, {0.45, 0.45}, {12.08, 12.08}, {19.39, 10.42}};
```

Then, X_1 and X_2 are given by:

```
{X1, X2} = Transpose[data];
```

There are $n = 42$ pairs. An unbiased estimator of the central moment $\mu_{2,1}$ is given by the h-statistic $h_{2,1}$. Using it yields the following estimate of $\mu_{2,1}$:

```
HStatistic[{2, 1}, 42] /. s_{i_, j_} :> X1^i.X2^j
```

$h_{2,1} \to 752.787$

An alternative estimator of $\mu_{2,1}$ is the sample central moment $m_{2,1}$:

```
m21 = SampleCentralToPowerSum[{2, 1}]
```

$$m_{2,1} \to \frac{2\, s_{0,1}\, s_{1,0}^2}{n^3} - \frac{2\, s_{1,0}\, s_{1,1}}{n^2} - \frac{s_{0,1}\, s_{2,0}}{n^2} + \frac{s_{2,1}}{n}$$

Unfortunately, $m_{2,1}$ is a biased estimator, and it yields a different estimate here:

```
m21 /. {s_{i_, j_} :> X1^i.X2^j, n -> 42}
```

$m_{2,1} \to 699.87$

The `CentralMoment` function in *Mathematica*'s Statistics package also implements the biased estimator $m_{2,1}$:

```
<< Statistics`MultiDescriptiveStatistics`
CentralMoment[data, {2, 1}]
```

699.87

7.3 Moments of Moments

7.3 A Getting Started

Let $(X_1, \ldots, X_n)$ denote a random sample of size n drawn from the population random variable X. Because $(X_1, \ldots, X_n)$ are random variables, it follows that a statistic like the sample central moment m_r is itself a random variable, with its own distribution and its own population moments. Suppose we want to find the expectation of m_2. Since $E[m_2]$ is just the first raw moment of m_2, we can denote this problem by $\acute{\mu}_1(m_2)$. Similarly, we might want to find the population variance of $\acute{m}_1$. Since $\text{Var}(\acute{m}_1)$ is just the second central moment of $\acute{m}_1$, we can denote this problem by $\mu_2(\acute{m}_1)$. Or, we might want to find the fourth cumulant of m_3, which we denote by $\kappa_4(m_3)$. In each of these cases, we are finding a population moment of a sample moment, or, for short, a *moment of a moment*.

The problem of *moments of moments* has attracted a prolific literature containing many beautiful formulae. Such formulae are listed over pages and pages of tables in reference texts and journals. Sometimes these tables contain errors; sometimes one induces errors oneself by typing them in incorrectly; sometimes the desired formula is simply not available and deriving the solution oneself is cumbersome and tricky. Some authors have devoted years to this task! The tools presented in this chapter change all that: they enable one to generate any desired formula, usually in just a few seconds, without even having to worry about typing it in incorrectly.

Although the problem of *moments of moments* has produced a long and complicated literature, conceptually the problem is rather simple. Let $p(s)$ denote any symmetric rational polynomial expressed in terms of power sums s_r (§7.1 B). Our goal is to find the population moments of p, and to express the answer in terms of the population moments of X. Let $\acute{\mu}_r(p)$, $\mu_r(p)$ and $\kappa_r(p)$ denote, respectively, the r^{th} raw moment, central moment and cumulant of p. In each case, we can present the solution in terms of raw moments $\acute{\mu}_i(X)$ of the population of X, or central moments $\mu_i(X)$ of the population of X, or cumulants $\kappa_i(X)$ of the population of X. As such, the problem can be expressed in 9 different ways:

$$\left.\begin{matrix}\acute{\mu}_r(p)\\ \mu_r(p)\\ \kappa_r(p)\end{matrix}\right\}\quad \text{in terms of} \quad \left\{\begin{matrix}\acute{\mu}_i(X)\\ \mu_i(X)\\ \kappa_i(X)\end{matrix}\right.$$

Consequently, **mathStatica** offers 9 functions to tackle the problem of *moments of moments*, as shown in Table 1.

function	*description*
`RawMomentToRaw[r, p]`	$\acute{\mu}_r(p)$ in terms of $\acute{\mu}_i(X)$
`RawMomentToCentral[r, p]`	$\acute{\mu}_r(p)$ in terms of $\mu_i(X)$
`RawMomentToCumulant[r, p]`	$\acute{\mu}_r(p)$ in terms of $\kappa_i(X)$
`CentralMomentToRaw[r, p]`	$\mu_r(p)$ in terms of $\acute{\mu}_i(X)$
`CentralMomentToCentral[r, p]`	$\mu_r(p)$ in terms of $\mu_i(X)$
`CentralMomentToCumulant[r, p]`	$\mu_r(p)$ in terms of $\kappa_i(X)$
`CumulantMomentToRaw[r, p]`	$\kappa_r(p)$ in terms of $\acute{\mu}_i(X)$
`CumulantMomentToCentral[r, p]`	$\kappa_r(p)$ in terms of $\mu_i(X)$
`CumulantMomentToCumulant[r, p]`	$\kappa_r(p)$ in terms of $\kappa_i(X)$

Table 1: Moments of moments functions

For instance, consider the function `CentralMomentToRaw[r, p]`:

- the term `CentralMoment` indicates that we wish to find $\mu_r(p)$; *i.e.* the r^{th} central moment of p;
- the term `ToRaw` indicates that we want the answer expressed in terms of raw moments $\acute{\mu}_i$ of the population of X.

These functions nest common operators such as:

- the expectation operator: $E[p] = \acute{\mu}_1(p) =$ `RawMomentTo?[1, p]`
- the variance operator: $\mathrm{Var}(p) = \mu_2(p) =$ `CentralMomentTo?[2, p]`

There is often more than one correct way of thinking about these problems. For example, the expectation $E[p^3]$ can be thought of as either $\acute{\mu}_1(p^3)$ or as $\acute{\mu}_3(p)$. Endnote 3 provides more detail on the ___`ToCumulant` functions; it should be carefully read before using them.

⊕ ***Example 8:*** Checking if the Unbiased Estimators Really Are Unbiased

We are now equipped to test, for instance, whether the unbiased estimators introduced in §7.2 *really are* unbiased. In §7.2 C, we obtained the polykay $k_{\{2,4\}}$ in terms of power sums:

```
p = PolyK[{2, 4}]
```

$$k_{\{2,4\}} \to (6\,s_1^6 - 18\,n\,s_1^4\,s_2 + (30 - 27\,n + 15\,n^2)\,s_1^2\,s_2^2 + (60 - 60\,n + 21\,n^2 - 3\,n^3)\,s_2^3 + (-40 + 36\,n + 4\,n^2)\,s_1^3\,s_3 + (-120 + 100\,n - 24\,n^2 - 4\,n^3)\,s_1\,s_2\,s_3 + (40 - 10\,n - 10\,n^2 + 4\,n^3)\,s_3^2 + (60 - 20\,n - 15\,n^2 - n^3)\,s_1^2\,s_4 + (-60 + 45\,n - 10\,n^2 + n^4)\,s_2\,s_4 + (24 - 42\,n + 12\,n^2 + 6\,n^3)\,s_1\,s_5 + (-4\,n + 7\,n^2 - 2\,n^3 - n^4)\,s_6) / ((-5 + n)\,(-4 + n)\,(-3 + n)\,(-2 + n)\,(-1 + n)\,n)$$

This statistic is meant to have the property that $E[p] = \kappa_2\,\kappa_4$. Since $E[p] = \acute{\mu}_1(p)$, we will use the `RawMomentTo?[1, p]` function; moreover, since the answer is desired in terms of cumulants, we use the suffix `ToCumulant`:

```
RawMomentToCumulant[1, p⟦2⟧]
```

$\kappa_2\,\kappa_4$

... so all is well. Similarly, we can check the h-statistics. Here is the 4th h-statistic in terms of power sums:

```
p = HStatistic[4]
```

$$h_4 \to \frac{-3\,s_1^4 + 6\,n\,s_1^2\,s_2 + (9 - 6\,n)\,s_2^2 + (-12 + 8\,n - 4\,n^2)\,s_1\,s_3 + (3\,n - 2\,n^2 + n^3)\,s_4}{(-3 + n)\,(-2 + n)\,(-1 + n)\,n}$$

This is meant to have the property that $E[p] = \mu_4$. And …

```
RawMomentToCentral[1, p⟦2⟧]
```

μ_4

… all is well. ■

⊕ ***Example 9:*** The Variance of the Sample Mean $\acute{m}_1$

Step (i): Express $\acute{m}_1$ in terms of power sums: trivially, we have $\acute{m}_1 = \frac{s_1}{n}$.

Step (ii): Since $\text{Var}(\acute{m}_1) = \mu_2(\frac{s_1}{n})$, the desired solution is:

```
CentralMomentToCentral[2, s1/n]
```

$$\frac{\mu_2}{n}$$

where μ_2 denotes the population variance. This is just the well-known result that the variance of the sample mean is $\text{Var}(X)/n$. ■

⊕ ***Example 10:*** The Variance of m_2

Step (i): Convert m_2 into power sums (§7.1 B):

```
m2 = SampleCentralToPowerSum[2][[2]]
```

$$-\frac{s_1^2}{n^2} + \frac{s_2}{n}$$

Step (ii): Since $\text{Var}(m_2) = \mu_2(m_2)$, the desired solution is:

```
CentralMomentToCentral[2, m2]
```

$$-\frac{(-3+n)\ (-1+n)\ \mu_2^2}{n^3} + \frac{(-1+n)^2\ \mu_4}{n^3}$$

⊕ ***Example 11:*** Mean Square Error of Two Estimators

Which is the better estimator of μ_2^2: (a) the square of the second h-statistic h_2^2, or (b) the polyache $h_{\{2,2\}}$?

Solution: We know that the polyache $h_{\{2,2\}}$ is an unbiased estimator of μ_2^2, while h_2^2 is a biased estimator of μ_2^2. But bias is not everything: variance is also important. The *mean square error* of an estimator is a measure that takes account of both bias and variance, defined by $\text{MSE}(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]$, where $\hat{\theta}$ denotes the estimator, and θ is the true parameter value (see Chapter 9 for more detail). For this particular problem, the two estimators are $\overline{\theta} = h_2^2$ and $\tilde{\theta} = h_{\{2,2\}}$:

```
θ̄ = HStatistic[2][[2]]^2
θ̃ = PolyH[{2, 2}][[2]]
```

$$\frac{(-s_1^2 + n\ s_2)^2}{(-1+n)^2\ n^2}$$

$$\frac{s_1^4 - 2\ n\ s_1^2\ s_2 + (3 - 3\ n + n^2)\ s_2^2 + (-4 + 4\ n)\ s_1\ s_3 + (n - n^2)\ s_4}{(-3+n)\ (-2+n)\ (-1+n)\ n}$$

If we let $p = (\hat{\theta} - \theta)^2$, then $\text{MSE}(\hat{\theta}) = E[p] = \acute{\mu}_1(p)$, so the mean square error of each estimator is (in terms of central moments):

```
MSE[θ̄] = RawMomentToCentral[1, (θ̄ - μ₂²)²];

MSE[θ̃] = RawMomentToCentral[1, (θ̃ - μ₂²)²];
```

Now consider the ratio of the mean square errors of the two estimators. We are interested to see whether this ratio is greater than or smaller than 1. If it is always greater than 1, then the polykay $\tilde{\theta} = h_{\{2,2\}}$ is the strictly preferred estimator:

```
rat = MSE[θ̄] / MSE[θ̃] // Factor
```

$$((-3+n)\,(-2+n)\,(-630\,\mu_2^4 + 885\,n\,\mu_2^4 - 507\,n^2\,\mu_2^4 + 159\,n^3\,\mu_2^4 - 31\,n^4\,\mu_2^4 + 4\,n^5\,\mu_2^4 + 560\,\mu_2\,\mu_3^2 - 840\,n\,\mu_2\,\mu_3^2 + 520\,n^2\,\mu_2\,\mu_3^2 - 168\,n^3\,\mu_2\,\mu_3^2 + 24\,n^4\,\mu_2\,\mu_3^2 + 420\,\mu_2^2\,\mu_4 - 690\,n\,\mu_2^2\,\mu_4 + 430\,n^2\,\mu_2^2\,\mu_4 - 138\,n^3\,\mu_2^2\,\mu_4 + 30\,n^4\,\mu_2^2\,\mu_4 - 4\,n^5\,\mu_2^2\,\mu_4 - 35\,\mu_4^2 + 60\,n\,\mu_4^2 - 42\,n^2\,\mu_4^2 + 12\,n^3\,\mu_4^2 - 3\,n^4\,\mu_4^2 - 56\,\mu_3\,\mu_5 + 104\,n\,\mu_3\,\mu_5 - 72\,n^2\,\mu_3\,\mu_5 + 24\,n^3\,\mu_3\,\mu_5 - 28\,\mu_2\,\mu_6 + 64\,n\,\mu_2\,\mu_6 - 48\,n^2\,\mu_2\,\mu_6 + 16\,n^3\,\mu_2\,\mu_6 - 4\,n^4\,\mu_2\,\mu_6 + \mu_8 - 3\,n\,\mu_8 + 3\,n^2\,\mu_8 - n^3\,\mu_8))\,/\,(2\,(-1+n)^2\,n^2\,(-66\,\mu_2^4 + 51\,n\,\mu_2^4 - 17\,n^2\,\mu_2^4 + 2\,n^3\,\mu_2^4 + 48\,\mu_2\,\mu_3^2 - 28\,n\,\mu_2\,\mu_3^2 + 4\,n^2\,\mu_2\,\mu_3^2 + 36\,\mu_2^2\,\mu_4 - 36\,n\,\mu_2^2\,\mu_4 + 14\,n^2\,\mu_2^2\,\mu_4 - 2\,n^3\,\mu_2^2\,\mu_4 - 6\,\mu_4^2 + 5\,n\,\mu_4^2 - n^2\,\mu_4^2))$$

This expression seems too complicated to immediately say anything useful about it, so let us consider an example. If the population is $N(\mu, \sigma^2)$ with pdf $f(x)$:

```
f = 1/(σ √(2π)) Exp[-(x - μ)²/(2 σ²)];
domain[f] = {x, -∞, ∞} && {μ ∈ Reals, σ > 0};
```

… then the first 8 central moments of the population are:

```
mgfc = Expect[e^(t (x-μ)), f];
cm = Table[μᵢ → D[mgfc, {t, i}] /. t → 0, {i, 8}]
```

$$\{\mu_1 \to 0,\ \mu_2 \to \sigma^2,\ \mu_3 \to 0,\ \mu_4 \to 3\,\sigma^4,\ \mu_5 \to 0,\ \mu_6 \to 15\,\sigma^6,\ \mu_7 \to 0,\ \mu_8 \to 105\,\sigma^8\}$$

so the ratio becomes:

```
rr = rat /. cm // Factor
```

$$\frac{(-3+n)\,(-2+n)\,n\,(3+n)\,(1+2\,n)}{2\,(-1+n)^2\,(3+3\,n-4\,n^2+n^3)}$$

Figure 1 shows that this ratio is always greater than 1, irrespective of σ, so the polyache is strictly preferred, at least for this distribution.

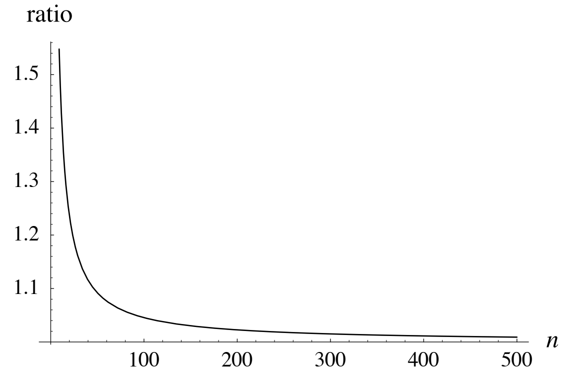

Fig. 1: $\frac{\text{MSE}(\bar{\theta})}{\text{MSE}(\tilde{\theta})}$ as a function of n, for the Normal distribution

We plot for $n > 9$ because the *moments of moments* functions are well-defined only for $n > w$, where w is the weight of the statistic. ■

7.3 B Product Moments

Product moments (multivariate moments) were introduced in §6.2 B and §6.2 D. We are interested here in expressions such as:

$$\begin{aligned}\acute{\mu}_{r,s}(p_a, p_b) &= E[p_a^r\, p_b^s]\\ \mu_{r,s}(p_a, p_b) &= E\big[(p_a - E[p_a])^r\ (p_b - E[p_b])^s\big]\\ \kappa_{r,s}(p_a, p_b)\end{aligned}$$

where each p_i is a symmetric polynomial in power sums s_i. All of **mathStatica**'s *moment of moment* functions generalise to neatly handle product moments — given $\mu_r(p)$, simply think of r and p as lists.

⊕ ***Example 12:*** Find the Covariance Between the Sample Moments m_2 and m_3

Step (i): Express m_2 and m_3 in terms of power sums:

```
m2 = SampleCentralToPowerSum[2][[2]];
m3 = SampleCentralToPowerSum[3][[2]];
```

Step (ii): *Example 13* of Chapter 6 showed that $\text{Cov}(m_2, m_3)$ is just the product moment $\mu_{1,1}(m_2, m_3)$. Thus, the solution is:

```
CentralMomentToCentral[{1, 1}, {m2, m3}]
```

$$-\frac{2\ (-2+n)\ (-1+n)\ (-5+2\,n)\ \mu_2\,\mu_3}{n^4} + \frac{(-2+n)\ (-1+n)^2\ \mu_5}{n^4}$$

7.3 C Cumulants of k-statistics

Following the work of Fisher (1928), the cumulants of k-statistics have received great attention, for which two reasons are proffered. First, it is often claimed that the cumulants of the k-statistics yield much more compact formulae than other derivations. This is not really true. Experimentation with the *moment of moment* functions shows that $\acute{\mu}_r(\mathrm{k}_i)$ is just as compact as $\kappa_r(\mathrm{k}_i)$, provided both results are expressed *in terms of cumulants*. In this sense, there is nothing special about cumulants of k-statistics per se; the raw moments of the k-statistics are just as compact. Second, Fisher showed how the cumulants of the k-statistics can be derived using a combinatoric method, in contrast to the algebraic method *du jour*. While Fisher's combinatorial approach is less burdensome algebraically, it is tricky and finicky, which can easily lead to errors. Indeed, with **mathStatica**, one can show that even after 70 years, a reference bible such as Stuart and Ord (1994) *still* contains errors in its listings of cumulants of k-statistics; examples are provided below. **mathStatica** uses an internal algebraic approach because (i) this is general, safe and secure, and (ii) the burdensome algebra ceases to be a constraint when you can get a computer to do all the dreary work for you. It is perhaps a little ironic then that modern computing technology has conceptually taken us full circle back to the work of Pearson (1902), Thiele (1903), and 'Student' (1908).

In this section, we will make use of the following k-statistics:

```
k2 = KStatistic[2]⟦2⟧;
k3 = KStatistic[3]⟦2⟧;
```

Here are the first four cumulants of k_2, namely $\kappa_r(\mathrm{k}_2)$ for $r = 1, 2, 3, 4$:

```
CumulantMomentToCumulant[1, k2]
```

$$\kappa_2$$

```
CumulantMomentToCumulant[2, k2]
```

$$\frac{2\,\kappa_2^2}{-1+n} + \frac{\kappa_4}{n}$$

```
CumulantMomentToCumulant[3, k2]
```

$$\frac{8\,\kappa_2^3}{(-1+n)^2} + \frac{4\,(-2+n)\,\kappa_3^2}{(-1+n)^2\,n} + \frac{12\,\kappa_2\,\kappa_4}{(-1+n)\,n} + \frac{\kappa_6}{n^2}$$

```
CumulantMomentToCumulant[4, k2]
```

$$\frac{48\,\kappa_2^4}{(-1+n)^3} + \frac{96\,(-2+n)\,\kappa_2\,\kappa_3^2}{(-1+n)^3\,n} + \frac{144\,\kappa_2^2\,\kappa_4}{(-1+n)^2\,n} + \frac{8\,(6-9\,n+4\,n^2)\,\kappa_4^2}{(-1+n)^3\,n^2} + \frac{32\,(-2+n)\,\kappa_3\,\kappa_5}{(-1+n)^2\,n^2} + \frac{24\,\kappa_2\,\kappa_6}{(-1+n)\,n^2} + \frac{\kappa_8}{n^3}$$

Next, we derive the product cumulant $\kappa_{3,1}(k_3, k_2)$, expressed in terms of cumulants, as obtained by David and Kendall (1949, p. 433). This takes less than 2 seconds to solve on our reference computer:

```
CumulantMomentToCumulant[{3, 1}, {k3, k2}]
```

$$\frac{1296\, n\, (-12+5n)\, \kappa_2^4\, \kappa_3}{(-2+n)^2\, (-1+n)^3} + \frac{324\, (164-136\, n+29\, n^2)\, \kappa_2\, \kappa_3^3}{(-2+n)^2\, (-1+n)^3} + \frac{648\, (137-126\, n+29\, n^2)\, \kappa_2^2\, \kappa_3\, \kappa_4}{(-2+n)^2\, (-1+n)^3} + \frac{108\, (-390+543\, n-257\, n^2+41\, n^3)\, \kappa_3\, \kappa_4^2}{(-2+n)^2\, (-1+n)^3\, n} + \frac{108\, (110-122\, n+33\, n^2)\, \kappa_2^3\, \kappa_5}{(-2+n)^2\, (-1+n)^3} + \frac{54\, (-564+842\, n-421\, n^2+71\, n^3)\, \kappa_3^2\, \kappa_5}{(-2+n)^2\, (-1+n)^3\, n} + \frac{54\, (316-340\, n+93\, n^2)\, \kappa_2\, \kappa_4\, \kappa_5}{(-2+n)\, (-1+n)^3\, n} + \frac{54\, (178-220\, n+63\, n^2)\, \kappa_2\, \kappa_3\, \kappa_6}{(-2+n)\, (-1+n)^3\, n} + \frac{9\, (103-134\, n+49\, n^2)\, \kappa_5\, \kappa_6}{(-1+n)^3\, n^2} + \frac{54\, (-23+12\, n)\, \kappa_2^2\, \kappa_7}{(-2+n)\, (-1+n)^2\, n} + \frac{27\, (22-31\, n+11\, n^2)\, \kappa_4\, \kappa_7}{(-1+n)^3\, n^2} + \frac{9\, (-26+17\, n)\, \kappa_3\, \kappa_8}{(-1+n)^2\, n^2} + \frac{45\, \kappa_2\, \kappa_9}{(-1+n)\, n^2} + \frac{\kappa_{11}}{n^3}$$

⊕ ***Example 13:*** Find the Correlation Coefficient Between k_2 and k_3

Solution: If ρ_{XY} denotes the correlation coefficient between random variables X and Y, then by definition:

$$\rho_{XY} = \frac{E\big[(X-E[X])\,(Y-E[Y])\big]}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \quad \text{so that} \quad \rho_{k_2\, k_3} = \frac{E\big[(k_2-\kappa_2)\,(k_3-\kappa_3)\big]}{\sqrt{\mu_2(k_2)\,\mu_2(k_3)}}$$

The solution (expressed here in terms of cumulants) is thus:

$$\frac{\texttt{RawMomentToCumulant[1, (k2 - }\kappa_2\texttt{) (k3 - }\kappa_3\texttt{)]}}{\sqrt{\texttt{CentralMomentToCumulant[2, k2] CentralMomentToCumulant[2, k3]}}}$$

$$\frac{\frac{6\,\kappa_2\,\kappa_3}{-1+n} + \frac{\kappa_5}{n}}{\sqrt{\left(\frac{2\,\kappa_2^2}{-1+n} + \frac{\kappa_4}{n}\right)\left(\frac{6\,n\,\kappa_2^3}{(-2+n)\,(-1+n)} + \frac{9\,\kappa_3^2}{-1+n} + \frac{9\,\kappa_2\,\kappa_4}{-1+n} + \frac{\kappa_6}{n}\right)}}$$

Since $E\big[(X - E[X])\,(Y - E[Y])\big] = \mu_{1,1}(X, Y)$, we could alternatively derive the numerator as:

```
CentralMomentToCumulant[{1, 1}, {k2, k3}]
```

$$\frac{6\,\kappa_2\,\kappa_3}{-1+n} + \frac{\kappa_5}{n}$$

which gives the same answer. ■

○ *Product Cumulants*

These tools can be used to check the tables of product cumulants provided in texts such as Stuart and Ord (1994), which in turn are based on Fisher's (1928) results (with corrections). We find full agreement, except for $\kappa_{2,2}(\mathrm{k}_3, \mathrm{k}_2)$ (Stuart and Ord, equation 12.70) which we correctly obtain as:

```
CumulantMomentToCumulant[{2, 2}, {k3, k2}]
```

$$\frac{288\,n\,\kappa_2^5}{(-2+n)\,(-1+n)^3} + \frac{288\,(-23+10\,n)\,\kappa_2^2\,\kappa_3^2}{(-2+n)\,(-1+n)^3} +$$
$$\frac{360\,(-7+4\,n)\,\kappa_2^3\,\kappa_4}{(-2+n)\,(-1+n)^3} + \frac{36\,(160-155\,n+38\,n^2)\,\kappa_3^2\,\kappa_4}{(-2+n)\,(-1+n)^3\,n} +$$
$$\frac{36\,(93-103\,n+29\,n^2)\,\kappa_2\,\kappa_4^2}{(-2+n)\,(-1+n)^3\,n} +$$
$$\frac{24\,(202-246\,n+71\,n^2)\,\kappa_2\,\kappa_3\,\kappa_5}{(-2+n)\,(-1+n)^3\,n} + \frac{2\,(113-154\,n+59\,n^2)\,\kappa_5^2}{(-1+n)^3\,n^2} +$$
$$\frac{6\,(-131+67\,n)\,\kappa_2^2\,\kappa_6}{(-2+n)\,(-1+n)^2\,n} + \frac{3\,(117-166\,n+61\,n^2)\,\kappa_4\,\kappa_6}{(-1+n)^3\,n^2} +$$
$$\frac{6\,(-27+17\,n)\,\kappa_3\,\kappa_7}{(-1+n)^2\,n^2} + \frac{37\,\kappa_2\,\kappa_8}{(-1+n)\,n^2} + \frac{\kappa_{10}}{n^3}$$

By contrast, Fisher (1928) and Stuart and Ord (1994) give the coefficient of the $\kappa_2^3\,\kappa_4$ term as $\frac{72\,(-23+14\,n)}{(-2+n)\,(-1+n)^3}$; for the $\kappa_2^2\,\kappa_3^2$ term: $\frac{144\,(-44+19\,n)}{(-2+n)\,(-1+n)^3}$. There is also a small typographic error in Stuart and Ord equation 12.66, $\kappa_{2,1}(\mathrm{k}_4, \mathrm{k}_2)$, though this is correctly stated in Fisher (1928).

⊕ ***Example 14:*** Show That Fisher's (1928) Solution for $\kappa_{2,2}(\mathrm{k}_3, \mathrm{k}_2)$ Is Incorrect

If we can show that Fisher's solution is wrong for one distribution, it must be wrong generally. In this vein, let $X \sim \text{Bernoulli}(\frac{1}{2})$, so that $X^i = X$ for any integer i. Hence, $s_1 = s_2 = s_3 = Y \sim \text{Binomial}(n, \frac{1}{2})$ (*cf. Example 21* of Chapter 4). Recall that the k-statistics k_2 and k_3 were defined above in terms of power sums s_i. We can now replace all power sums s_i in k_2 and k_3 with the random variable Y:

```
K2 = k2 /. s_i_ → y
```

$$\frac{n\,y - y^2}{(-1+n)\,n}$$

```
K3 = k3 /. s_i_ → y
```

$$\frac{n^2\,y - 3\,n\,y^2 + 2\,y^3}{(-2+n)\,(-1+n)\,n}$$

where random variable $Y \sim \text{Binomial}(n, \frac{1}{2})$, with pmf $g(y)$:

```
g  = Binomial[n, y] p^y (1 - p)^(n-y) /. p → 1/2;
domain[g] = {y, 0, n} && {n > 0, n ∈ Integers} && {Discrete};
```

We now want to calculate the product cumulant $\kappa_{2,2}(\text{K}_3, \text{K}_2)$ directly, when $Y \sim \text{Binomial}(n, \frac{1}{2})$. The product cumulant $\kappa_{2,2}$ can be expressed in terms of product raw moments as follows:

```
κ22 = CumulantToRaw[{2, 2}]
```

$$\kappa_{2,2} \to -6\,\acute{\mu}_{0,1}^2\,\acute{\mu}_{1,0}^2 + 2\,\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}^2 + 8\,\acute{\mu}_{0,1}\,\acute{\mu}_{1,0}\,\acute{\mu}_{1,1} - 2\,\acute{\mu}_{1,1}^2 - 2\,\acute{\mu}_{1,0}\,\acute{\mu}_{1,2} + 2\,\acute{\mu}_{0,1}^2\,\acute{\mu}_{2,0} - \acute{\mu}_{0,2}\,\acute{\mu}_{2,0} - 2\,\acute{\mu}_{0,1}\,\acute{\mu}_{2,1} + \acute{\mu}_{2,2}$$

as given in Cook (1951). Here, each term $\acute{\mu}_{r,s}$ denotes $\acute{\mu}_{r,s}(\text{K}_3, \text{K}_2) = E[\text{K}_3^r\,\text{K}_2^s]$, and hence can be evaluated with the `Expect` function. In the next input, we calculate each of the expectations that we require:

```
Ω = κ22⟦2⟧ /. μ́_{r_,s_} :→ Expect[K3^r K2^s, g] // Simplify
```

$$\frac{496 - 405\,n + 124\,n^2 - 18\,n^3 + n^4}{32\,(-2+n)\,(-1+n)^3\,n^3}$$

Hence, Ω is the value of $\kappa_{2,2}(\text{k}_3, \text{k}_2)$ when $X \sim \text{Bernoulli}(\frac{1}{2})$.

Fisher (1928) obtains, for any distribution whose moments exist, that $\kappa_{2,2}(\text{k}_3, \text{k}_2)$ is:

$$\text{Fisher} = \frac{288\,n\,\kappa_2^5}{(-2+n)\,(-1+n)^3} + \frac{144\,(-44+19\,n)\,\kappa_2^2\,\kappa_3^2}{(-2+n)\,(-1+n)^3} + \frac{72\,(-23+14\,n)\,\kappa_2^3\,\kappa_4}{(-2+n)\,(-1+n)^3} + \frac{36\,(160-155\,n+38\,n^2)\,\kappa_3^2\,\kappa_4}{(-2+n)\,(-1+n)^3\,n} + \frac{36\,(93-103\,n+29\,n^2)\,\kappa_2\,\kappa_4^2}{(-2+n)\,(-1+n)^3\,n} + \frac{24\,(202-246\,n+71\,n^2)\,\kappa_2\,\kappa_3\,\kappa_5}{(-2+n)\,(-1+n)^3\,n} + \frac{2\,(113-154\,n+59\,n^2)\,\kappa_5^2}{(-1+n)^3\,n^2} + \frac{6\,(-131+67\,n)\,\kappa_2^2\,\kappa_6}{(-2+n)\,(-1+n)^2\,n} + \frac{3\,(117-166\,n+61\,n^2)\,\kappa_4\,\kappa_6}{(-1+n)^3\,n^2} + \frac{6\,(-27+17\,n)\,\kappa_3\,\kappa_7}{(-1+n)^2\,n^2} + \frac{37\,\kappa_2\,\kappa_8}{(-1+n)\,n^2} + \frac{\kappa_{10}}{n^3};$$

Now, when $X \sim \text{Bernoulli}(\frac{1}{2})$, with pmf $f(x)$:

```
f = 1/2;  domain[f] = {x, 0, 1} && {Discrete};
```

... the cumulant generating function is:

```
cgf = Log[Expect[e^(t x), f]]
```

$$\text{Log}\left[\frac{1}{2}\,(1 + e^t)\right]$$

and so the first 10 cumulants are:

```
κlis = Table[κ_r → D[cgf, {t, r}] /. t → 0, {r, 10}]
```

$$\left\{\kappa_1 \to \frac{1}{2},\ \kappa_2 \to \frac{1}{4},\ \kappa_3 \to 0,\ \kappa_4 \to -\frac{1}{8},\ \kappa_5 \to 0,\ \kappa_6 \to \frac{1}{4},\ \kappa_7 \to 0,\ \kappa_8 \to -\frac{17}{16},\ \kappa_9 \to 0,\ \kappa_{10} \to \frac{31}{4}\right\}$$

… so Fisher's solution becomes:

```
Fsol = Fisher /. κlis // Simplify
```

$$\frac{496 - 405\,n + 124\,n^2 - 72\,n^3 + 28\,n^4}{32\,(-2 + n)\,(-1 + n)^3\,n^3}$$

which is *not* equal to Ω derived above. Hence, Fisher's (1928) solution must be incorrect. How does **mathStatica** fare? When $X \sim \text{Bernoulli}(\frac{1}{2})$, our solution is:

```
CumulantMomentToCumulant[{2, 2}, {k3, k2}]
  /. κlis // Simplify
```

$$\frac{496 - 405\,n + 124\,n^2 - 18\,n^3 + n^4}{32\,(-2 + n)\,(-1 + n)^3\,n^3}$$

which *is* identical to Ω, as we would expect. How big is the difference between the two solutions? The following output shows that, when $X \sim \text{Bernoulli}(\frac{1}{2})$, Fisher's solution is at least 28 times too large, and as much as 188 times too large:

```
Fsol/Ω /. n → {11, 20, 50, 100, 500, 1000000000} // N
```

{188.172, 68.0391, 38.7601, 32.8029, 28.882, 28.}

This comparison is only valid for n greater than the weight w of $\kappa_{2,2}(k_3, k_2)$, where $w = 10$ here. Weights are defined in the next section. ■

7.4 Augmented Symmetrics and Power Sums

7.4 A Definitions and a Fundamental Expectation Result

This section does not strive to solve new problems; instead, it describes the building blocks upon which unbiased estimators and *moments of moments* are built. Primarily, it deals with converting expressions such as the three-part sum $\sum_{i \neq j \neq k} X_i X_j^2 X_k^2$ into one-part sums such as $\sum_{i=1}^{n} X_i^r$. The former are called *augmented symmetrics functions*, while the latter are one-part symmetrics, more commonly known as *power sums*. Formally, as per §7.1 B, the r^{th} *power sum* is defined as

$$s_r = \sum_{i=1}^{n} X_i^r, \qquad r = 1, 2, \ldots \tag{7.12}$$

Further, let $A_{\{a, b, c, \ldots\}}$ denote an augmented symmetric function of the variates. For example,

$$A_{\{3, 2, 2, 1\}} = \sum_{i \neq j \neq k \neq m} X_i^3 X_j^2 X_k^2 X_m^1 \tag{7.13}$$

where each index in the four-part sum ranges from 1 to n. For any list of positive integers t, the *weight* of A_t is $w = \Sigma t$, while the *order*, or number of parts, is the dimension of t, which we denote by ρ. For instance, $A_{\{3, 2, 2, 1\}}$ has weight 8, and order 4. For convenience, one can notate $A_{\{3, 2, 2, 1, 1, 1, 1\}}$ as $A_{\{3, 2^2, 1^4\}}$ corresponding to an 'extended form' and 'condensed form' notation, respectively. Many authors would denote $A_{\{3, 2^2, 1^4\}}$ by the expression $[3\, 2^2\, 1^4]$; unfortunately, this notation is ill-suited to *Mathematica* where [ ] notation is already 'taken'.

This section provides tools that enable one to:

(i) express an augmented symmetric function in terms of power sums; that is, find function f such that $A_t = f(s_1, s_2, \ldots, s_w)$—each term in f will be *isobaric* (have the same weight w);

(ii) express products of power sums (*e.g.* $s_1\, s_2\, s_3$) in terms of augmented symmetric functions.

Past attempts: Considerable effort has gone into deriving tables to convert between symmetric functions and power sums. This includes the work of O'Toole (1931, weight 6, contains errors), Dwyer (1938, weight 6), Sukhatme (1938, weight 8), and Kerawala and Hanafi (1941, 1942, 1948) for $w = 9$ through 12 (errors). David and Kendall (1949) independently derived a particularly neat set of tables up to weight 12, though this set is also not free of error, though a later version, David *et al.* (1966, weight 12) appears to be correct. With **mathStatica**, we can extend the analysis far beyond weight 12, and derive correct solutions of even weight 20 in just a few seconds.

○ *Augmented Symmetrics to Power Sums*

The **mathStatica** function `AugToPowerSum` converts a given augmented symmetric function into power sums. Here we find $[3\ 2^3] = A_{\{3,2^3\}}$ in terms of power sums:

```
AugToPowerSum[{3, 2, 2, 2}]
```

$$A_{\{3,2,2,2\}} \to s_2^3\, s_3 - 3\, s_2\, s_3\, s_4 - 3\, s_2^2\, s_5 + 3\, s_4\, s_5 + 2\, s_3\, s_6 + 6\, s_2\, s_7 - 6\, s_9$$

The integers in `AugToPowerSum[{3, 2, 2, 2}]` do not need to be any particular order. In fact, one can even use 'condensed-form' notation:[4]

```
AugToPowerSum[{3, 2^3}]
```

$$A_{\{3,2,2,2\}} \to s_2^3\, s_3 - 3\, s_2\, s_3\, s_4 - 3\, s_2^2\, s_5 + 3\, s_4\, s_5 + 2\, s_3\, s_6 + 6\, s_2\, s_7 - 6\, s_9$$

Standard tables also list the related monomial symmetric functions, though these are generally less useful than the augmented symmetrics. Using condensed form notation, the *monomial symmetric* $M_{\{a^\alpha, b^\beta, c^\chi, \ldots\}}$ is defined by:

$$M_{\{a^\alpha, b^\beta, c^\chi, \ldots\}} = \frac{A_{\{a^\alpha, b^\beta, c^\chi, \ldots\}}}{\alpha!\, \beta!\, \chi! \ldots} . \tag{7.14}$$

mathStatica provides a function to express monomial symmetric functions in terms of power sums. Here is $M_{\{3,2^3\}}$:

```
MonomialToPowerSum[{3, 2^3}]
```

$$M_{\{3,2,2,2\}} \to \frac{1}{6}\, s_2^3\, s_3 - \frac{1}{2}\, s_2\, s_3\, s_4 - \frac{1}{2}\, s_2^2\, s_5 + \frac{s_4\, s_5}{2} + \frac{s_3\, s_6}{3} + s_2\, s_7 - s_9$$

○ *Power Sums to Augmented Symmetrics*

The **mathStatica** function `PowerSumToAug` converts products of power sums into augmented symmetric functions. For instance, to find $s_1\, s_2^3$ in terms of $A_{\{\}}$:

```
PowerSumToAug[{1, 2, 2, 2}]
```

$$s_1\, s_2^3 \to A_{\{7\}} + 3\, A_{\{4,3\}} + 3\, A_{\{5,2\}} + A_{\{6,1\}} + 3\, A_{\{3,2,2\}} + 3\, A_{\{4,2,1\}} + A_{\{2,2,2,1\}}$$

Here is an example with weight 20 and order 20. It takes less than a second to find the solution, but many pages to display the result:

```
PowerSumToAug[{1^20}]; // Timing
```

```
{0.93 Second, Null}
```

Like most other converter functions, these functions also allow one to specify ones own notation. Here, we keep 's' to denote power sums, but change the $A_{\{\}}$ terms to $\lambda_{\{\}}$:

```
PowerSumToAug[{3, 2, 3}, s, λ]
```

$s_2\, s_3^2 \to \lambda_{\{8\}} + 2\, \lambda_{\{5,3\}} + \lambda_{\{6,2\}} + \lambda_{\{3,3,2\}}$

○ *A Fundamental Expectation Result*

A fundamental expectation result (Stuart and Ord (1994), Section (12.5)) is that

$$E[A_{\{a,b,c,\ldots\}}] \;=\; \acute{\mu}_a\, \acute{\mu}_b\, \acute{\mu}_c \cdots \quad\times\quad n\,(n-1)\cdots(n-\rho+1) \tag{7.15}$$

where, given A_t, the symbol ρ denotes the number of elements in the list t. This result is important because it lies at the very heart of both the unbiased estimation of population moments, and the *moments of moments* literature (see §7.4 B and C below). As a simple illustration, suppose we want to prove that $\acute{m}_r$ is an unbiased estimator of $\acute{\mu}_r$ (7.4): to do so, we first express $\acute{m}_r = \frac{s_r}{n} = \frac{A_{\{r\}}}{n}$ so that we have an expression in $A_{\{r\}}$, and then apply (7.15) to yield $E\big[\acute{m}_r\big] = \frac{1}{n}\, E[A_{\{r\}}] = \acute{\mu}_r$.

We can implement (7.15) in *Mathematica* as follows:

```
ExpectAug[t_] :=
  (Thread[μ́_t] /. List → Times) ∏_{i=0}^{Length[t]-1} (n - i)
```

Thus, the expectation of say $A_{\{2,2,3\}}$ is given by:

```
ExpectAug[{2, 2, 3}]
```

$(-2+n)\;(-1+n)\;n\,\acute{\mu}_2^2\,\acute{\mu}_3$

⊕ ***Example 15:*** An Unbiased Estimator of $\acute{\mu}_3\,\acute{\mu}_4$

In *Example 6*, we found an unbiased estimator of $\acute{\mu}_3\,\acute{\mu}_4$ by converting to cumulants, and then finding an unbiased estimator for each cumulant by using polykays. It is much easier to apply the expectation theorem (7.15) directly, from which it follows immediately that an unbiased estimator of $\acute{\mu}_3\,\acute{\mu}_4$ is $\frac{A_{\{3,4\}}}{n(n-1)}$, where $A_{\{3,4\}}$ is given by:

```
AugToPowerSum[{3, 4}]
```

$A_{\{3,4\}} \to s_3\, s_4 - s_7$

as we found in *Example 6*. ■

7.4 B Application 1: Understanding Unbiased Estimation
Augmented Symmetrics → Power Sums

Let us suppose that we wish to find an unbiased estimator of $\kappa_2\,\kappa_1\,\kappa_1$ from first principles. Now, $\kappa_2\,\kappa_1\,\kappa_1$ can be written in terms of raw moments:

```
z1 = Times @@
   Map[CumulantToRaw[#]⟦2⟧ &, {2, 1, 1}] // Expand
```

$$-\acute{\mu}_1^4 + \acute{\mu}_1^2\,\acute{\mu}_2$$

We have just found the coefficients of the polykay $k_{\{2,1,1\}}$ in terms of so-called *Wishart Tables* (see Table 1 of Wishart (1952) or Appendix 11 of Stuart and Ord (1994)). To obtain the inverse relation in such tables, use RawToCumulant instead of CumulantToRaw. In ListForm notation (noting that the order of the terms is now reversed), we have:

```
z2 = ListForm[z1, μ́]
```

$$\acute{\mu}[\{1,1,2\}] - \acute{\mu}[\{1,1,1,1\}]$$

By the fundamental expectation result (7.15), an unbiased estimator of z1 (or z2) is:

$$\texttt{z3 = z2 /. } \acute{\mu}\texttt{[x\_]} :\to \frac{\texttt{AugToPowerSum[x]⟦2⟧}}{\prod_{\texttt{i=0}}^{\texttt{Length[x]-1}} (\texttt{n - i})} \quad \texttt{// Factor}$$

$$\frac{-s_1^4 + 3\,s_1^2\,s_2 + n\,s_1^2\,s_2 - n\,s_2^2 - 2\,s_1\,s_3 - 2\,n\,s_1\,s_3 + 2\,n\,s_4}{(-3+n)\,(-2+n)\,(-1+n)\,n}$$

This result is identical to PolyK[{2,1,1}], other than the ordering of the terms.

7.4 C Application 2: Understanding Moments of Moments
Products of Power Sums → Augmented Symmetrics

We wish to find an exact method for finding moments of sampling distributions in terms of population moments, which is what the *moments of moments* functions do, but now from first principles. Equation (7.15) enables one to find the expectation of a moment, by implementing the following three steps:

(i) convert that moment into power sums,
(ii) convert the power sums into augmented symmetrics, and
(iii) then apply the fundamental expectation result (7.15) using ExpectAug.

For example, to find $E[m_4]$, we first convert m_4 into power sums s_i:

```
m4 = SampleCentralToPowerSum[4]⟦2⟧
```

$$-\frac{3\,s_1^4}{n^4} + \frac{6\,s_1^2\,s_2}{n^3} - \frac{4\,s_1\,s_3}{n^2} + \frac{s_4}{n}$$

Then, after converting into `ListForm`, convert into augmented symmetrics:

```
z1 = ListForm[m4, s] /. s[x_] :> PowerSumToAug[x][[2]]
```

$$\frac{A_{\{4\}}}{n} - \frac{4\,(A_{\{4\}} + A_{\{3,1\}})}{n^2} + \frac{6\,(A_{\{4\}} + A_{\{2,2\}} + 2\,A_{\{3,1\}} + A_{\{2,1,1\}})}{n^3} - \frac{3\,(A_{\{4\}} + 3\,A_{\{2,2\}} + 4\,A_{\{3,1\}} + 6\,A_{\{2,1,1\}} + A_{\{1,1,1,1\}})}{n^4}$$

We can now apply the fundamental expectation result (7.15):

```
z2 = z1 /. A_t_ :> ExpectAug[t] // Simplify
```

$$-\frac{1}{n^3}\Big((-1+n)\Big(3\,(6-5\,n+n^2)\,\acute{\mu}_1^4 - 6\,(6-5\,n+n^2)\,\acute{\mu}_1^2\,\acute{\mu}_2 + (9-6\,n)\,\acute{\mu}_2^2 + 4\,(3-3\,n+n^2)\,\acute{\mu}_1\,\acute{\mu}_3 - (3-3\,n+n^2)\,\acute{\mu}_4\Big)\Big)$$

This output is identical to that given by `RawMomentToRaw[1, m4]`, except that the latter does a better job of ordering the terms of the resulting polynomial.

7.5 Exercises

1. Which of the following are rational, integral, algebraic symmetric functions?

(i) $\sum_{i=1}^{n} X_i^2$ (ii) $\left(\sum_{i=1}^{n} X_i\right)^2$ (iii) $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ (iv) $\sqrt{\sum_{i=1}^{n}(X_i-\overline{X})^2}$

(v) $\text{h}_2\, m_3^2$ (vi) h_2 / m_3^2 (vii) $\text{h}_2 + m_3^2$ (viii) $\sqrt{\text{h}_2\, m_3^2}$

2. Express each of the following in terms of power sums:

(i) $\sum_{i=1}^{n} X_i^4$ (ii) $\left(\sum_{i=1}^{n} X_i\right)^2$ (iii) $m_3 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^3$

(iv) $\text{k}_4\, m_2^3$ (v) $(\text{h}_3-5)^2$ (vi) $\sum_{i=1}^{n}\left((X_i-\overline{X})^3\,(Y_i-\overline{Y})^2\right)$

3. Find an unbiased estimator of: (i) μ_3 (ii) $\mu_3^2\,\mu_2$ (iii) κ_{13} ☺ (iv) the sixth factorial moment. Verify that each solution is, in fact, an unbiased estimator.

4. Solve the following: (i) $\text{Var}(m_4)$ (ii) $E\left[\sum_{i=1}^{n} X_i^2\right]$ (iii) $E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right]$

(iv) $\kappa_4(\text{k}_2)$ (v) $\mu_{3,2}(\text{h}_2, \text{h}_3)$.

5. Let $(X_1, \ldots, X_n)$ denote a random sample of size n drawn from $X \sim \text{Lognormal}(\mu, \sigma)$. Let $Y = \sum_{i=1}^{n} X_i$. Find the first 4 raw moments of Y.

6. Find the covariance between $\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$ and $\frac{1}{n}\sum_{i=1}^{n} X_i$. What can be said about the covariance if the population is symmetric?